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1001. (a) The factor theorem tells us that $\left(x+\frac{2}{3}\right)$ must be a factor. Any constant multiple of this must also be factor, so $3\left(x+\frac{2}{3}\right) \equiv (3x+2)$ is a factor.

(b) Taking out this factor,

$$\begin{aligned} & 6x^3+37x^2-41x-42 \\ \equiv\ & (3x+2)(2x^2+11x-21) \\ \equiv\ & (3x+2)(2x-3)(x+7). \end{aligned}$$

—— Alternative Method ——

Using polynomial long division,

$$\begin{array}{r|l} & 2x^2+11x-21 \\ 3x+2 & 6x^3+37x^2-41x-42 \\ & -\,6x^3\ \ -4x^2 \\ \hline & \quad 33x^2-41x \\ & -\,33x^2-22x \\ \hline & \qquad -\,63x-42 \\ & \qquad\ \ 63x+42 \\ \hline & \qquad\qquad 0 \end{array}$$

Factorising the quotient $2x^2+11x-21$ gives $(3x+2)(2x-3)(x+7)$.

1002. Without the information, $\mathbb{P}(X \geq 3)=\frac{4}{6}$. With it, the possibility space is restricted to $\{2,3,5\}$. Over this space, $\mathbb{P}(X \geq 3)=\frac{2}{3}$. So, the information given does not affect $\mathbb{P}(X \geq 3)$.

1003. Subtracting y^2 and separating the variables,

$$\begin{aligned} & x\frac{dy}{dx}+y^2=1 \\ \Longrightarrow\ & x\frac{dy}{dx}=1-y^2 \\ \Longrightarrow\ & \frac{1}{1-y^2}\frac{dy}{dx}=\frac{1}{x} \end{aligned}$$

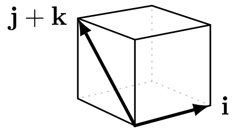
The above holds if $x \neq 0$ and $1-y^2 \neq 0$.

1004. Defining $x=\log_a b$ and $y=\log_b a$, we can rewrite as $a^x=b$ and $b^y=a$. Substituting the former into the latter gives $(a^x)^y=a$, which, by an index law, is $a^{xy}=a$. Hence, $xy=1$. Rewriting in terms of a and b, $\log_a b \times \log_b a \equiv 1$. QED.

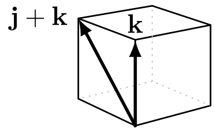
1005. The endpoints of the line are $(0,2)$ and $(4,-2)$. Evaluating x^2+y^2 at these points gives 4 and 20. Comparing these to 10, one endpoint lies inside the circle, and one lies outside. Hence, the line must intersect the circle.

1006. Using Pythagoras, $|BD|=\sqrt{2}$. So, $\triangle BXD$ has side lengths $\left(1,1,\sqrt{2}\right)$. These are a Pythagorean triple, so angle BXD is a right angle.

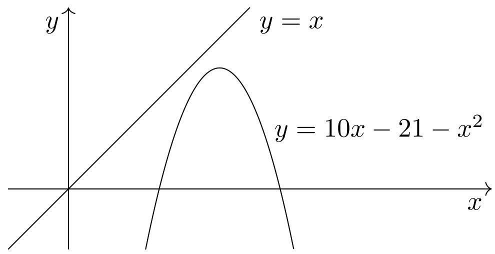
1007. (a) $90°$. Since $\mathbf{i}$ is perpendicular to both $\mathbf{j}$ and $\mathbf{k}$, it is perpendicular to any linear combination of them.



(b) $45°$. The vector $\mathbf{i}$ is not involved, so this is a 2D problem: we are looking for the angle between the edge of a square and its diagonal.



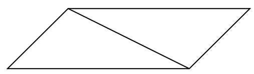
1008. Solving for intersections between $y=10x-21-x^2$ and $y=x$, we get $-x^2+9x-21=0$, which has discriminant $\Delta=81-4\cdot 21=-3<0$. So, the parabola $y=10x-21-x^2$ does not intersect $y=x$. Therefore, since $y=10x-21-x^2$ is a negative parabola, it must be below $y=x$.



So, if $y=10x-21-x^2$, then $y<x$.

1009. (a) This is true. It's a standard result: the SSS condition for congruency.

(b) This is true. A parallelogram can be divided up into two triangles along one diagonal, and these triangles then satisfy SSS. Since their constituent triangles are congruent, so are the parallelograms.



1010. A general quadratic graph is $y=ax^2+bx+c$, for $a \neq 0$. For stationary points, we require $2ax+b=0$. This gives $x=-\frac{b}{2a}$. If the stationary point is on the y axis, then $x=0$, so $b=0$. This is equivalent to saying that the quadratic has no term in x. □

—— Alternative Method ——

Assume, for a contradiction, that such a graph has a non-zero term in x. Then it is $y=ax^2+bx+c$, where $a,b \neq 0$. Setting the derivative to zero, $2ax+b=0 \implies x=-\frac{b}{2a}$. This is non-zero, so the vertex of the quadratic is not on the y axis. This is a contradiction. So, if a quadratic graph $y=\mathrm{f}(x)$ has a stationary point on the y axis, then it has no term in x. □

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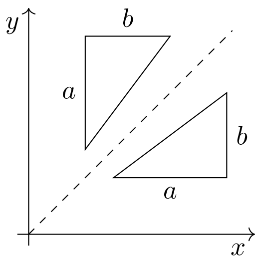
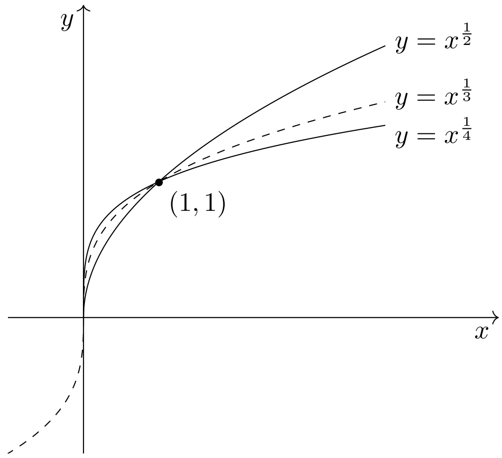
1011. The statement is true. This can be seen as a direct algebraic result of the chain rule, or, equivalently, a graphical argument can be used. Reflection in the line $y = x$ transforms $\frac{dy}{dx}$ to $\frac{dy}{dx}$ by switching x and y, and such a reflection reciprocates gradient triangles:



1012. Taking the square root of the second equation, we get $y = \pm 5 - x$. Substituting this into the first equation gives $x^2 \pm 5 - x = 7$. These are two quadratics, $x^2 - x - 12 = 0$ and $x^2 - x - 2 = 0$. Solving these gives four solution points:

$$(4, -9), (-3, -2), (2, 3), (-1, 6).$$

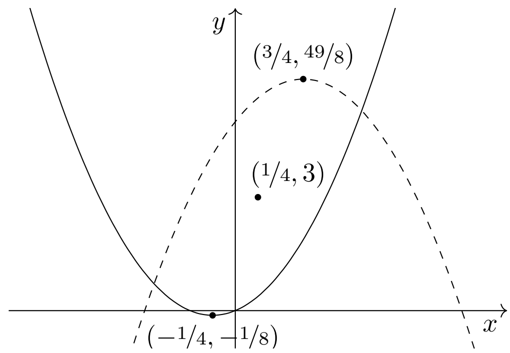
1013. The graph $y = x^{\frac{1}{2}}$ is a half-parabola, and $y = x^{\frac{1}{4}}$ is akin, though more snub-nosed. The shape of $y = x^{\frac{1}{3}}$ is between the two, but it is defined for negative x as well:



Nota Bene

Here, I am using the word "akin" to mean "of the same family as". In everyday English, we might say "similar", but this has a specific mathematical meaning.

1014. Completing the square gives $y = 2(x + 1/4)^2 - 1/8$ and $y = -2(x-3/4)^2 + 49/8$. The vertices (minimum and maximum respectively) are at $(-1/4, -1/8)$ and $(3/4, 49/8)$. The midpoint of these is $(1/4, 3)$, which is the centre of rotation.



So, the rotation is by 180° around $(1/4, 3)$.

1015. (a) $(-2, 2) \cap [1, 3] = [1, 2)$,
(b) $\big(\mathbb{R} \setminus [-2, 2]\big) \cap [1, 3] = (2, 3]$,
(c) $[0, 2] \cap [1, 3] = [1, 2]$.

1016. A sample is a subset of the population, which is smaller than the population and is intended to be representative of it. A census is a sample, but it is the biggest possible sample, consisting of (as far as is possible) the entire population. The UK census occurs every 10 years, in years ending with 1.

1017. Multiples of 3 can be considered as an AP. Using the standard formula for the partial sum of an AP,

$$\begin{aligned} S_n &= \tfrac{n}{2}\big(2a + (n-1)d\big) \\ &= \tfrac{100}{2}(2 \cdot 3 + 99 \cdot 3) \\ &= 15150. \end{aligned}$$

1018. A' is the complement of A, i.e. the negation, in set terms, of A: everything that is not in A. The symbol $\notin$ means "is not an element of". So, both statements are true, by definition.

1019. The numerator is a difference of two squares. So, we divide top and bottom by $p - q \neq 0$:

$$\begin{aligned} &\lim_{p \to q} \frac{p^2 - q^2}{p - q} \\ \equiv\ &\lim_{p \to q} (p + q). \end{aligned}$$

Taking the limit, this is $2q$.

Nota Bene

The above limit is another way of expressing the first-principles differentiation of the curve $y = x^2$, considering p and q as two values of x. The above proves that

$$\left.\frac{dy}{dx}\right|_{x=q} = 2q.$$

1020. The terms of the sum are the odd numbers, which are the shaded/unshaded chevrons. So, the sum is the number of squares in the grid, which is n^2.

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1021. Multiplying out, $y = ax^3 - abx^2$. Setting the derivative to zero for SPs,

$$\begin{aligned} 3ax^2 - 2abx &= 0 \\ \implies ax(3x - 2b) &= 0 \end{aligned}$$

If $a = 0$, then the curve is $y = 0$, which cannot have a maximum at $(2, 4)$. So, $x = 0, \frac{2b}{3}$. Since the graph has a maximum at $x = 2$, we know that $\frac{2b}{3} = 2$. Hence, $b = 3$. Substituting $(2, 4)$ gives $a = -1$.

1022. (a) A "particle" is an object of negligible size. It is considered mathematically as existing at a zero-dimensional point.

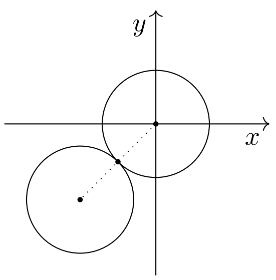
(b) A "rod" is a rigid object, whose size in two of its dimensions is negligible. Mathematically, a one-dimensional line segment.

(c) A "projectile" is a particle accelerating under gravity. Other forces are negligible. On Earth, a projectile accelerates at g ms^{-2} vertically downwards.

— Nota Bene —

The word "negligible" is a useful one in modelling. It suggests that its user is sophisticated enough not to make the erroneous assumption (often made) that e.g. size is physically zero. Instead, the user assumes that size is physically close enough to zero to allow a mathematical setting to zero.

1023. This is true. The curves are circles of radius $\sqrt{2}$, and, by Pythagoras, their centres are $2\sqrt{2}$ away from each other. So, they are tangent at $(-1, -1)$.



1024. Rearranging and factorising gives

$$\begin{aligned} &\cos x(\cos^2 x - 1) = 0 \\ \implies &\cos x(\cos x + 1)(\cos x - 1) = 0 \\ \implies &\cos x = -1, 0, 1. \end{aligned}$$

So, the solution is $x \in \{0°, 90°, 180°, 270°\}$.

1025. The height of the right-hand rectangle is n, so the lighter, large triangle has area $\frac{1}{2}n^2$. The n darker, small triangles each have area $\frac{1}{2}$. Hence, the total area is $\frac{1}{2}n^2 + \frac{1}{2}n = \frac{1}{2}n(n + 1)$ as required. QED.

1026. (a) Solving simultaneously for intersections gives $x = -3/2, 1$ for the first curve, and $x = 3, 11/2$ for the second. The line has gradient $-1/2$, so we are looking for a tangent gradient of 2. This occurs at $x = 1$ for the first graph, and $x = 3$ for the second.

(b) The (shortest) distance between the curves runs along this normal line. So, we need only find the distance between the relevant points $(1, 1)$ and $(3, 0)$. By Pythagoras, the distance is $\sqrt{5}$.

1027. The sphere is smooth, so a contact force cannot have a frictional component acting tangentially to the surface. Hence, any contact force can only be a reaction force, acting perpendicular to the surface. Lines perpendicular to the surface of a sphere are (extended) radii, which pass through the centre.

1028. Quoting the first-principles formula,

$$\frac{d}{dx}\big(k\,\text{f}(x)\big) = \lim_{h\to 0} \frac{k\,\text{f}(x+h) - k\,\text{f}(x)}{h}.$$

The factor k is constant, so it can be taken out of the limit, giving

$$k \lim_{h\to 0} \frac{\text{f}(x+h) - \text{f}(x)}{h}$$

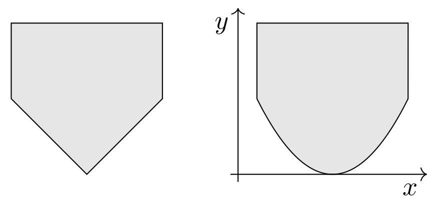
The limit is then $\text{f}'(x)$, by definition. Hence,

$$\frac{d}{dx}(k\,\text{f}(x)) = k\,\text{f}'(x), \text{ as required.}$$

1029. The relevant diagonals are between $(-4, 8)$ and $(12, 0)$, and between $(4, 9)$ and $(0, -1)$. These have equations $x + 2y = 12$ and $5x - 2y = 2$. Solving simultaneously gives $(7/3, 29/6)$.

— Nota Bene —

The term "convex", applied to polygons, has the same underlying meaning as when applied to curves. For a curve, when "convex" is used on its own (not as "convex up" or "convex down", as used by some authors) it means "convex up", i.e. that the shape up above the curve is convex:



Convex polygon   Convex curve

For both a convex polygon and a convex curve, all diagonals/chords lie wholly within the shape. As to why the region *above* the curve came to define the simple definition "convex", well, that's just the way things have come to be! It's useful though, because "concave" curves then look like the rock above a cave:

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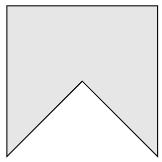
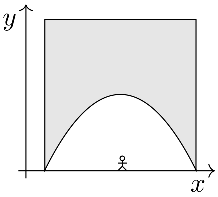


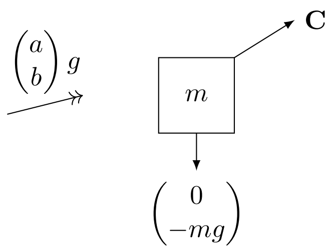
Concave polygon



Concave curve

1030. (a) $a = -1$ is a counterexample to the rightwards implication, so this does not hold.
(b) The cube function is one-to-one, so this holds.
(c) $a = -1$ is a counterexample to the rightwards implication, so this does not hold.

1031. Working in column vectors, model the passenger as follows:



The equation of motion is

$$\mathbf{C} + \begin{pmatrix} 0 \\ -mg \end{pmatrix} = m \begin{pmatrix} a \\ b \end{pmatrix} g.$$

This gives the contact force, in Newtons, as

$$\mathbf{C} = \begin{pmatrix} amg \\ (b+1)mg \end{pmatrix} = \begin{pmatrix} a \\ b+1 \end{pmatrix} mg.$$

1032. The gradient is $\frac{c}{a}$. So, the equation of the line is $y - b = \frac{c}{a}(x - a)$. Substituting $x = 0$ gives the y intercept as $y = b - c$, which is independent of a.

— Alternative Method —

The sequence $x = 2a$ to $x = a$ to $x = 0$ is an AP. This is a straight line, so the y values also follow an AP: $y = b + c$ to $y = b$ to $y = b - c$.

1033. The edge lengths are $x, 2x, 5x$. The surface area is

$$2\big(x^2 + 5x^2 + 10x^2\big) = 32x^2 = 1152.$$

Taking the positive square root gives $x = 6$. The volume is then $10x^3 = 2160 \text{ cm}^3$.

1034. The maximum possible value of $\mathbb{P}(X' \cap Y')$ is $\frac{1}{3}$, which is attained if $Y \subset X$. The minimum value is $1 - {}^2\!/_3 - {}^1\!/_4 = {}^1\!/_{12}$, which is attained when X and Y are mutually exclusive. So, the set of possible values for $\mathbb{P}(X' \cap Y')$ is $[{}^1\!/_{12}, {}^1\!/_3]$.

1035. (a) This is true. If $a = b$, then $x = a$ is also a root of $\mathrm{g}(x) = 0$, so $\mathrm{f}(a) = \mathrm{g}(a) = 0$.
(b) This is not true. If $a = b$, then $x = a$ is a root of $\mathrm{f}(x) = \mathrm{g}(x)$, but c could be another root of the same equation, not equal to a.

1036. Expanding the evaluations,

$$\begin{aligned} &(a^2 - 2a) - (0) = (4 + 4) - (0) \\ \implies\ & a^2 - 2a - 8 = 0 \\ \implies\ & a = 4, -2. \end{aligned}$$

1037. Writing a as $b^{\log_b a}$, we have a transformation

$$b^x \mapsto b^{x \log_b a}.$$

This is a replacement of x by $x \log_b a$, which is a stretch in the x direction. The scale factor is $\frac{1}{\log_b a}$, which may be simplified to $\log_a b$.

1038. The vacuum cleaner and the fan would be useless, as they work by propelling air. The machine-gun and the fire-extinguisher would be useful, as they both propel their contents (bullets or foam), and, by NIII, such propulsion involves a force exerted on the propelling entity. This force could be used, as per NII, to navigate back to the spacecraft.

— Nota Bene —

The vacuum cleaner and fan might, in fact, have some use, because you could always throw them!

1039. The sum of the interior angles of a 16-gon is $(16 - 2)\pi = 14\pi$. So, the mean angle is $\frac{7\pi}{8}$.

1040. This is an infinite geometric series, with first term $a = 1$ and common ratio $r = -\frac{1}{2}$. Hence, quoting the standard formula, which holds since $|r| < 1$, we have

$$S_\infty = \frac{a}{1 - r} = \frac{1}{1 - \big(-\frac{1}{2}\big)} = \frac{2}{3}.$$

1041. (a) After the first is picked, the probability that the second matches is $\mathbb{P}(\text{pair}) = \frac{1}{13}$.
(b) Given that they match, the individual socks are no longer relevant. There are seven days, and seven pairs, so the probability is simply $\mathbb{P}(\text{correct} \mid \text{pair}) = \frac{1}{7}$.

1042. Expanding and simplifying,

$$\begin{aligned} &\lim_{a \to 0} \frac{(1 + 3a + 3a^2 + a^3) - 1}{(1 + 4a + 6a^2 + 4a^3 + a^4) - 1} \\ =\ & \lim_{a \to 0} \frac{3a + 3a^2 + a^3}{4a + 6a^2 + 4a^3 + a^4} \\ =\ & \lim_{a \to 0} \frac{3 + 3a + a^2}{4 + 6a + 4a^2 + a^3} \\ =\ & \frac{3}{4}. \end{aligned}$$

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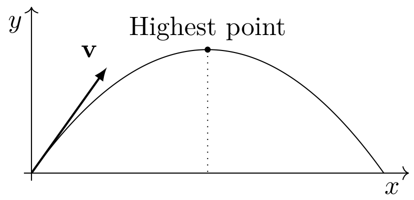
1043. The equation $\tan\theta = 1$ has another root $\theta = \frac{5\pi}{4}$ (and infinitely many others). Hence, statement (a) is false, disproved by the counterexample $\theta = \frac{5\pi}{4}$.

1044. (a) Since this extra data point lies exactly on the line of best fit, it will have contributed to an overestimation of the extent to which the data is linear. Unless the correlation happens to be perfect, removing it will reduce $|r|$ slightly.

(b) Removal will have no effect on the line of best fit, as the line of best fit always passes through the mean $(\bar{x}, \bar{y})$.

1045. The two lines are at angles $\arctan m_1$ and $\arctan m_2$ above the positive x axis. So, the angle between the lines is the difference between these: $\theta = \arctan m_2 - \arctan m_1$.

1046. In general, $x = v_x t$ and $y = v_y t - \frac{1}{2}gt^2$. Since x and t are related linearly, substituting gives the equation of the trajectory as $y = \mathrm{h}(x)$, where h is a negative quadratic function.



Any parabola is symmetrical about its vertex, as may be shown explicitly by completing the square to the form $y = a(x-b)^2 + c$. So, the trajectory is symmetrical about the highest point. □

1047. The statement is false. Integrating, we can see that $\mathrm{f}(x) = \mathrm{g}'(x) + c$, for some constant c. If $y = \mathrm{g}(x)$ is stationary at $x = \alpha$, then $\mathrm{g}'(\alpha) = 0$. However, if $c \neq 0$, then $\mathrm{f}(\alpha) \neq 0$, which means $\mathrm{f}(x) = 0$ doesn't have a root at $x = \alpha$.

1048. Since $\frac{d}{dx}(\tan x) = \sec^2 x$, we have

$$\int_0^{\frac{\pi}{4}} \sec^2 x \, dx = \Big[\tan x\Big]_0^{\frac{\pi}{4}} = 1 - 0 = 1.$$

1049. The possibility space has $6 \times 12 = 72$ outcomes, of which 6 are successful. So, the probability is $\frac{1}{12}$.

——— Alternative Method ———

Rolling the six-sided die first, all of the outcomes $\{1, 2, 3, 4, 5, 6\}$ are in the possibility space for the twelve-sided die. So, the probability is $\frac{1}{12}$.

1050. Multiplying top and bottom of the main fraction by x^2,

$$\begin{aligned} \frac{1}{1 - \frac{1}{x} + \frac{1}{x^2}} &= x \\ \Longrightarrow \frac{x^2}{x^2 - x + 1} &= x \\ \Longrightarrow x^2 &= x^3 - x^2 + x \\ \Longrightarrow x^3 - 2x^2 + x &= 0. \end{aligned}$$

Factorising this cubic,

$$\begin{aligned} x(x-1)^2 &= 0 \\ \Longrightarrow x &= 0, 1. \end{aligned}$$

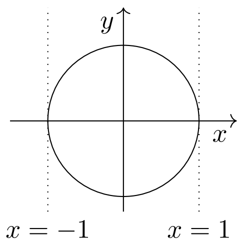
But $x = 0$ gives division by zero in the original equation, so the solution is $x = 1$.

——— Nota Bene ———

The "phantom root" is introduced in moving from the first to the second line. The main fraction in the first line is undefined for $x = 0$; the fraction in the second line has value 0 at $x = 0$.

We introduce no new roots in moving from the second to the third line, because $x^2 - x + 1$ has negative discriminant, and cannot be zero.

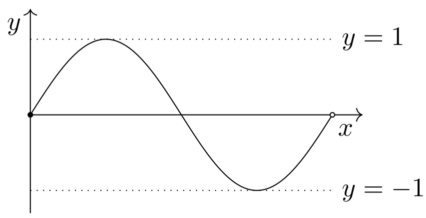
1051. A general point on the unit circle has coordinates $(\cos\theta, \sin\theta)$. So, we consider intersections of the unit circle and the line $x = k$. The only values of k with exactly one intersection are $k = \pm 1$:



With $|k| > 1$, $x = k$ will produce no intersections; with $|k| < 1$ it will produce two intersections.

——— Alternative Method ———

The graphs of $y = \cos x$ and $y = \pm 1$, with domain $[0, 360°)$, are as follows:



With $|k| > 1$, $y = k$ will produce no intersections; with $|k| < 1$ it will produce two intersections.

1052. The axis intercepts are $x = -6, 4$ and $y = \pm 4$.

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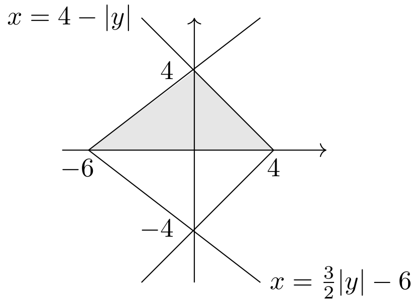


The area of the shaded triangle is $\frac{1}{2} \times 10 \times 4 = 20$. So, the area of the shaded kite is 40 square units.

1053. If $x - 2y$ is constant, then $x - 2y = c$, for constant c. Differentiating both sides of this equation gives $1 - 2\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = \frac{1}{2}$.

——— Alternative Method ———

If $x - 2y$ is constant, then it is stationary with respect to x (also with respect to y):

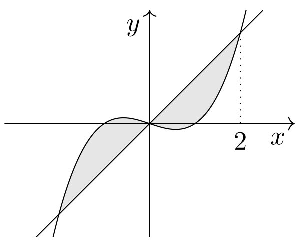
$$\begin{aligned} &\tfrac{d}{dx}(x - 2y) = 0 \\ \Longrightarrow\ &1 - 2\tfrac{dy}{dx} = 0 \\ \Longrightarrow\ &\tfrac{dy}{dx} = \tfrac{1}{2}. \end{aligned}$$

1054. The backwards implication $\Longleftarrow$ links them. If x is in $A \cap B$, then it must be in A, but the converse isn't true.

1055. (a) The integrand $x^3 - 4x$ represents the vertical displacement of $y = x^3 - x$ above $y = 3x$.

(b) The limits are the x values of intersections of the curves.

(c) There is another intersection, at $x = 0$, where the curves cross and the signed area changes from positive to negative. This has not been taken into account.



(d) The correct calculation is

$$\int_{-2}^{0} x^3 - 4x \, dx + \int_{0}^{2} 4x - x^3 \, dx = 8.$$

1056. (a) False. It may be even or odd.

(b) False. It may be even or odd.

(c) True.

1057. Taking out a factor of $(3x - 2)(x - 1)$,

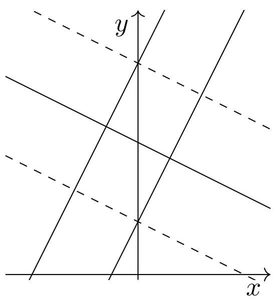
$$\begin{aligned} &(3x-2)(x-1)\big((3x-2)+(x-1)\big) = 0 \\ \Longrightarrow\ &(3x-2)(x-1)(4x-3) = 0 \\ \Longrightarrow\ &x = \tfrac{2}{3}, \tfrac{3}{4}, 1. \end{aligned}$$

1058. We can apply the operator $\frac{d}{dx}$ term by term:

$$\begin{aligned} &\tfrac{d}{dx}(x + y + 1) = 0 \\ \Longrightarrow\ &1 + \tfrac{dy}{dx} + 0 = 0 \\ \Longrightarrow\ &\tfrac{dy}{dx} = -1. \end{aligned}$$

1059. Since the quadratic factor is always positive, we can divide through by it, giving $x - 2 \geq 0$. Hence, the solution is $x \in [2, \infty)$.

1060. The fourth line must be of the form $x + 2y = k$.



Since its distance from $x + 2y = 5$ must be the same as the distance of $y = 2x + 1$ from $y = 2x + 4$, the possible equations are $x + 2y = 2$ and $x + 2y = 8$.

1061. Differentiating, $\frac{dy}{dx} = 2x$. So, the LHS is

$$\begin{aligned} &y\tfrac{dy}{dx} - 4x \\ =\ &(x^2 + 2)(2x) - 4x \\ =\ &2x^3 + 4x - 4x \\ =\ &2x^3, \text{ as required.} \end{aligned}$$

1062. Consider the boundary cases. At the lower end, the "hexagon" has a side of length 0 cm. At the upper end, the hexagon is regular, with all sides of length 10 cm. Neither of these boundary cases is attainable: one isn't a hexagon at all, and the other isn't irregular. So, l must be between 0 and 10, not inclusive. In set notation, $l \in (0, 10)$.

1063. Integrating, we get $\text{f}(x) - \text{g}(x) = 2x + c$. Since the RHS is linear and non-constant, it is zero at exactly one x value. Hence, so is the LHS. This is equivalent to saying that $\text{f}(x) = \text{g}(x)$ has exactly one root.

1064. We reshuffle the indices as follows:

$$\left(\sqrt[3]{a}\right)^x \equiv \left(a^{\frac{1}{3}}\right)^x \equiv a^{\frac{1}{3}x} \equiv \left(a^x\right)^{\frac{1}{3}} = z^{\frac{1}{3}}.$$

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— Nota Bene —

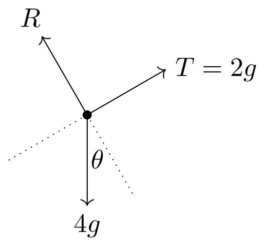
It is worth noting the above as an index law in its own right, proved via the standard index law $(x^a)^b = x^{ab}$. You can commute the order of nested indices in the following manner:

$$\left(x^a\right)^b \equiv \left(x^b\right)^a.$$

1065. Using the laws of logarithms:

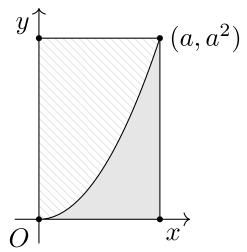
$$e^{3\ln a - \ln b} \equiv e^{\ln a^3 - \ln b} \equiv e^{\ln \frac{a^3}{b}} \equiv \tfrac{a^3}{b}.$$

1066. Since the string is inextensible, the hanging mass is also in equilibrium. Hence, the tension in the string is $T = 2g$. The force diagram for the 4 kg mass, with θ as the angle of inclination, is



Resolving along the string, $2g - 4g\sin\theta = 0$. So, $\sin\theta = \frac{1}{2}$. Since the angle of inclination must be acute, this gives $\theta = 30°$.

1067. Since three vertices are on the axes, one of the vertices must be the origin. Hence, without loss of generality we can place R in the positive quadrant:



With vertex (a, a^2) as labelled, the rectangle has area a^3. The area (shaded solid) beneath the curve is given by the integral

$$\int_0^a x^2\,dx = \left[\tfrac{1}{3}x^3\right]_0^a = \tfrac{1}{3}a^3.$$

Therefore, a third of the area is beneath the curve, which gives the required ratio $1 : 2$. □

1068. Since x^2 and y^2 are non-negative, we need only check that no sum of two squares from the set $\{0, 1, 4, 9, 16, 25, 36\}$ adds to 42.

$25 + 16 = 41 < 42$, so the only pairs large enough are $25 + 25 = 50$ or contain 36. But $36 + 4 = 40$ and $36 + 9 = 45$. Hence, $x^2 + y^2 = 42$ has no integer solutions. QED.

1069. Using 3D Pythagoras, we require

$$\tfrac{1}{7}\sqrt{2^2 + 3^2 + p^2} = 1.$$

This gives $p = \pm 6$.

1070. Setting $\Delta = 100$, we get $64 + 4a^2 = 100$, which has solution $a = \pm 3$.

1071. Pick the first card, without loss of generality. Then the probability that the remaining cards are from the same suit is

$$p = 1 \times \tfrac{12}{51} \times \tfrac{11}{50} \times \tfrac{10}{49} \times \tfrac{9}{48} = 0.00198 \text{ (3sf)}.$$

— Alternative Method —

There are ${}^{52}\mathrm{C}_5$ possible hands. There are 4 suits. For each suit, there are ${}^{13}\mathrm{C}_5$ possible flushes. So, the probability is

$$p = \frac{4 \times {}^{13}\mathrm{C}_5}{{}^{52}\mathrm{C}_5} = 0.00198 \text{ (3sf)}.$$

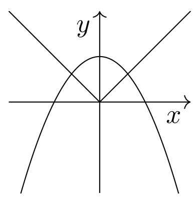
1072. This is a cubic in $x^{\frac{1}{3}}$. It factorises as

$$\begin{aligned} & x - 3x^{\frac{2}{3}} + 2x^{\frac{1}{3}} = 0 \\ \Longrightarrow\ & x^{\frac{1}{3}}(x^{\frac{2}{3}} - 3x^{\frac{1}{3}} + 2) = 0 \\ \Longrightarrow\ & x^{\frac{1}{3}}(x^{\frac{1}{3}} - 1)(x^{\frac{1}{3}} - 2) = 0 \\ \Longrightarrow\ & x^{\frac{1}{3}} = 0, 1, 2 \\ \Longrightarrow\ & x = 0, 1, 8. \end{aligned}$$

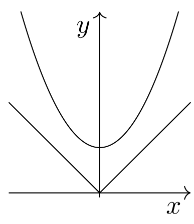
— Alternative Method —

Let $z = x^{\frac{1}{3}}$. This gives $z^3 - 3z^2 + 2z = 0$, which is a cubic. Using a polynomial solver or factorising, $z = 0, 1, 2$. Cubing these values, $x = 0, 1, 8$.

1073. (a) These graphs must intersect, since $y = 1 - x^2$ is a negative parabola with a maximum at $y = 1$.



(b) If these graphs are to intersect, we require $1 + x^2 = x$ to have at least one root. But it has discriminant $\Delta = -3 < 0$, so the graphs do not intersect.



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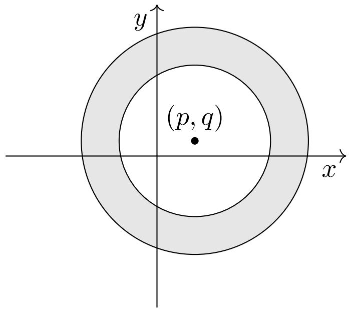
1074. Applying the iteration twice, we get $B_2 = 4a$, and then $B_3 = a(4a+1)$. So, we require

$$\begin{aligned} & a(4a+1) = 18 \\ \Longrightarrow\ & 4a^2 + a - 18 = 0 \\ \Longrightarrow\ & (a-2)(4a+9) = 0 \\ \Longrightarrow\ & a = 2, -\tfrac{9}{4}. \end{aligned}$$

1075. The fact that the annulus is centred on (p,q) is irrelevant. All we require is a disc of radius $\sqrt{b}$, with a disc of radius $\sqrt{a}$ removed from it.



The area is $\pi(b-a)$.

1076. The factor $(x^2 - c^2 - 1)$ is zero at $x = \pm\sqrt{1+c^2}$. Whatever the value of c (and whatever the values of the other constants) there is a division by zero at these x values, and the fraction is undefined.

1077. The statement holds. This is because, for there to be equilibrium, both a resultant force of zero and a resultant moment of zero are required.

1078. (a) The normal gradient is $-1/3$, so the tangent gradient at P is $m = 3$. The derivative is $3x^2$. So, we require

$$\begin{aligned} & 3x^2 = 3 \\ \Longrightarrow\ & x = \pm 1. \end{aligned}$$

The possible coordinates of P are $(\pm 1, \pm 1)$.

(b) Substituting the coordinates of P into the equation of the normal gives $k = \pm 4$.

1079. Any factorisation would have to be of the form

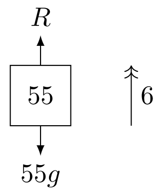
$$(x^2+1)(ax^3 + bx^2 + cx + d) \equiv 4x^5 + x + 1.$$

Multiplying out and equating coefficients of x^5, x^4, x^3, x^2 tells us that

$$\begin{aligned} a &= 4, \\ b &= 0, \\ a + c &= 0, \\ b + d &= 0. \end{aligned}$$

This requires $c = -4$ and $d = 0$. But this gives no constant term. Hence, no such factorisation is possible.

1080. The NIII pair of this force is the upwards force R exerted on the trapeze artist's hands. The force diagram is



Vertically, we have $R - 55g = 6 \cdot 55$, which gives 869 N upwards. So, the force on the bar is, by NIII, 869 N downwards.

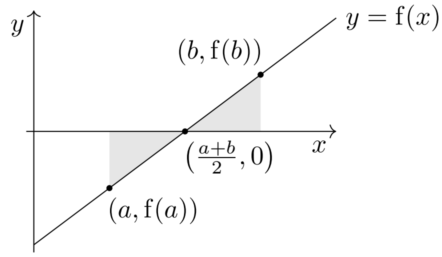
1081. The first equation is a quadratic in xy. Factorising gives $(4xy-1)^2 = 0$, so $4xy = 1$. Rearranging to $y = \frac{1}{4x}$, we substitute into the second equation:

$$\begin{aligned} & 4x + 4\tfrac{1}{4x} = 5 \\ \Longrightarrow\ & 4x^2 - 5x + 1 = 0 \\ \Longrightarrow\ & x = 1, \tfrac{1}{4}. \end{aligned}$$

So, the (x,y) solutions are $(1, 1/4)$ and $(1/4, 1)$.

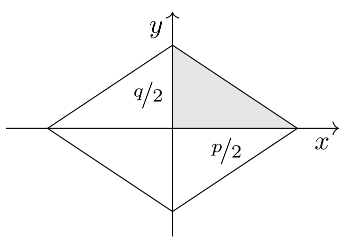
1082. The unit circle $x = \cos\theta$, $y = \sin\theta$ has area π. Stretching by scale factor a in the x direction and scale factor b in the y direction scales this area by ab, giving $A = \pi ab$.

1083. Since f is linear, and the signed area between the curve and the x axis is zero, the line $y = \mathrm{f}(x)$ must be symmetrical, in terms of y value, around the midpoint of the interval $[a,b]$.



At the midpoint, $\mathrm{f}\big(\frac{1}{2}(a+b)\big) = 0$.

1084. The diagonals of a rhombus are perpendicular, so a rhombus consists of four right-angled triangles with sides of length $p/2$ and $q/2$.



The total area is $A = 4 \cdot \frac{1}{2} \cdot \frac{p}{2} \cdot \frac{q}{2} \equiv \frac{1}{2}pq$.

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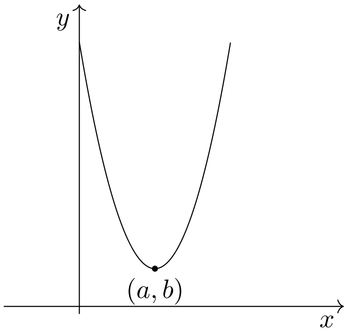
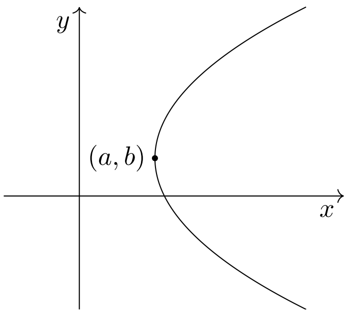
1085. (a) The graph $y - b = m(x-a)^2$ is a positive parabola of the form $y = f(x)$, with a vertex at (a, b), stretched in the y direction by scale factor m:



(b) The graph $\frac{1}{m}(y-b)^2 = (x-a)$ is a positive parabola of the form $x = f(y)$, with a vertex at (a, b), stretched in the x direction by scale factor $\frac{1}{m}$:



1086. Differentiating surface area $A = 2\pi rl$ with respect to time,

$$\frac{dA}{dt} = 2\pi r \frac{dl}{dt},$$

since radius r is a constant. We are told that the rate of change of length $\frac{dl}{dt}$ is constant. Hence, $\frac{dA}{dt}$ is also constant.

— Alternative Method —

The surface area is the product of circumference and length. Since only one of these dimensions is enlarging, the area scale factor is the same as the length scale factor. Hence, the rate of change of surface area is constant.

1087. This is equivalent to asking "Which point is closer to the origin?" The distances are $\sqrt{20}$ and $\sqrt{18}$, so $(3, 3)$ is closer.

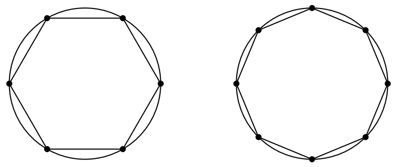
— Nota Bene —

Distance from a circle may be found via distance from the centre. This is a special case of the fact that the (shortest) distance from a point to a curve lies along the normal. In this case, the normal is a radius, which passes through the centre.

1088. For each log statement, there is an equivalent index statement. The reasons for the index statements are given afterwards.

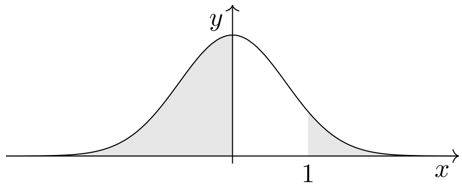
(a) $\log_a 1 = 0 \iff a^0 = 1$. The index statement is true because $1 = a^n \div a^n = a^{n-n} = a^0$.
(b) $\log_a a = 1 \iff a^1 = a$. The index statement is true by definition.
(c) $\log_a \sqrt{a} = \frac{1}{2} \iff a^{\frac{1}{2}} = \sqrt{a}$. Squaring the index statement gives $a^1 = a$, verifying that $a^{\frac{1}{2}}$ and $\sqrt{a}$ are equal.

1089. Since the diameter is the same, both shapes may be inscribed in the same circle. The edges of the $(2n+2)$-gon form a better approximation to the circle.



Hence, the $(2n+2)$-gon has the greater area.

1090. Solving $Z^2 > Z$ gives $Z \in (-\infty, 0) \cup (1, \infty)$. So, we require the following probability:



Using the normal facility on a calculator,

$$\begin{aligned} &\mathbb{P}(Z < 0) + \mathbb{P}(Z > 1) \\ &= 0.5 + 0.15865... \\ &= 0.659 \text{ (3sf)}. \end{aligned}$$

1091. Evaluating the derivative at $x = a$ gives gradient $2a$ at point (a, a^2). We use

$$y - y_1 = m(x - x_1).$$

The equation of the tangent is

$$y - a^2 = 2a(x - a).$$

Setting $y = 0$, $x = {}^a/_2$.

1092. (a) $1, 2, 4, 7, 11, ...$
(b) The first differences are $1, 2, 3, 4, ...$, and the second difference is 1. Hence, the sequence is quadratic, with nth term

$$u_n = \tfrac{1}{2}n^2 + pn + q.$$

The first two terms require $1 = {}^1/_2 + p + q$ and $2 = 2 + 2p + q$. Solving gives $p = -\frac{1}{2}$, $q = 1$. Hence, the ordinal formula is

$$u_n = \tfrac{1}{2}n^2 - \tfrac{1}{2}n + 1.$$

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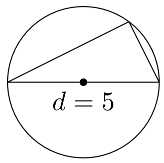
1093. The smallest circle which could possibly contain a triangle has diameter equal to its longest edge, in this case 5.



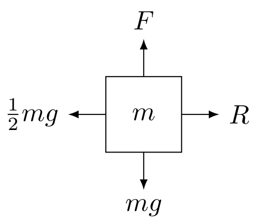
This circle is indeed possible. A $(3, 4, 5)$ triangle is right-angled. Hence, by the angle in a semicircle theorem, if the hypotenuse lies on the diameter, then the other vertex lies on the circumference. So, the smallest radius is $5/2$.

1094. This is a pair of linear simultaneous equations in $\sqrt{x}$ and y^2. So, let $X = \sqrt{x}$ and $Y = y^2$. The equations are

$$\begin{aligned} 2X + 3Y &= 18, \\ 3X - 2Y &= 1. \end{aligned}$$

Solving these gives $X = 3$, $Y = 4$. So $x = 9$ and $y = \pm 2$.

1095. The forces, with the left-hand face of the cube in contact with the wall, are as follows:



The least value of μ is attained when the block is in limiting equilibrium (on the point of slipping), so friction F is at $F_{\text{max}} = \mu R$. Vertical equilibrium gives $\mu R - mg = 0$. By horizontal equilibrium, $R = \frac{1}{2}mg$, so $\mu_{\text{min}} \cdot \frac{1}{2}mg - mg = 0$, which gives $\mu_{\text{min}} = 2$.

1096. The point $(1, a)$ must lie on the line, at $t = 0$. So $4 + a = 8$, and $a = 4$. Then the gradient of the line gives $-4 = \frac{b}{-2}$, so $b = 8$.

1097. A counterexample is $p, q, r, s = 1$. In that case, $p = q$, but the fraction in the right-hand equation is undefined.

1098. (a) The quartic, which is a biquadratic, factorises

$$\begin{aligned} &x^4 - 8x^2 + 16 \\ \equiv\ &(x^2 - 4)(x^2 - 4) \\ \equiv\ &(x + 2)^2(x - 2)^2. \end{aligned}$$

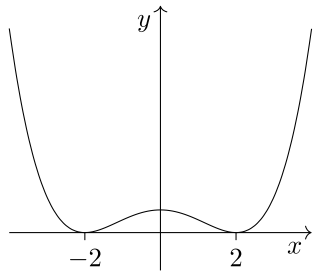
So, the double roots are at $x = \pm 2$.

(b) The derivative is $\frac{dy}{dx} = 4x^3 - 16x$. Evaluating this at $x = \pm 2$ yields

$$4(\pm 2)^3 - 16(\pm 2) = \pm 32 \mp 32 = 0.$$

Hence, the gradient is zero at the double roots.

(c) The curve is a positive quartic which is tangent to the x axis at $x = \pm 2$:



1099. The ratio information is $3(c - b) = 2(a - c)$. This rearranges to $5c - 2a = 3b$. If a is a multiple of 5, then the whole LHS is. Hence the RHS is too. This implies that b must be a multiple of 5. □

1100. Setting $\log_x y = a$, we have $x^a = y$. Raising both sides to the power $1/a$ gives $x = y^{1/a}$. Rewriting this as a log statement, $\log_y x = 1/a$. So, $\log_x y$ and $\log_y x$ are reciprocals. QED.

——— End of 11th Hundred ———

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